Affine varieties and coordinate rings

Def: An affine variety is an irreducible affine algebraic set.

So $\left\{\begin{matrix}\text{affine varieties} \\ \text{in } \mathbb{A}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals} \\ \text{in } k[x_1, \dots, x_n]\end{matrix}\right\}$

Functions on varieties

Let $V \subseteq \mathbb{A}^n$ be a variety (or alg. set). Let $\mathcal{F}(V, k)$ be the set of all functions $V \to k$. $\mathcal{F}(V, k)$ has a natural ring structure.

Def: $f \in \mathcal{F}(V, k)$ is a polynomial function or regular function on V if there is some $F \in k[x_1, \dots, x_n]$ s.t. $f = F|_V$.

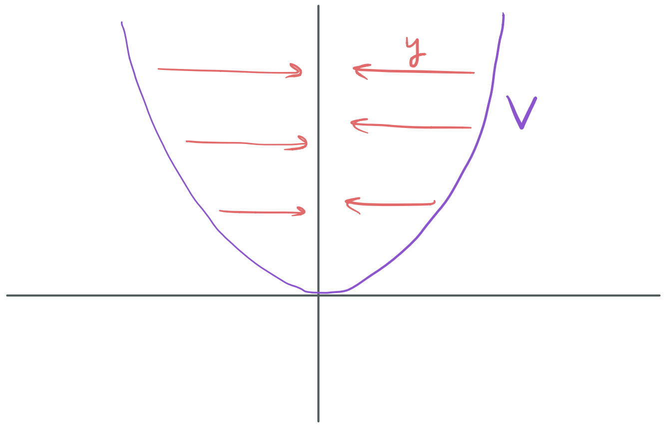
i.e. $\forall\ (a_1, \dots, a_n) \in V,\ F(a_1, \dots, a_n) = f(a_1, \dots, a_n)$.

Easy exercise: The set of regular functions on V is a subring of $\mathcal{F}(V, k)$.

Def: The ring of regular functions on V is called the coordinate ring of V. It's denoted $\Gamma(V)$.

Ex: 1.) $\Gamma(\mathbb{A}^n) = k[x_1, \dots, x_n]$.

2.) Let $V = V(y - x^2) = \{(t, t^2) \mid t \in k\}$



The function y outputs the y-coordinate. i.e. it is projection onto the y-axis.

The function x^2 is the same function on V.

3.) Consider $V(xy-1) \subseteq \mathbb{A}^2$. Is $\frac{1}{y}$ regular?

$xy = 1 \Rightarrow x = \frac{1}{y}$ so x and $\frac{1}{y}$ are the same function on $V(xy-1)$, so $\frac{1}{y}$ is regular.

In general, if $V \subseteq \mathbb{A}^n$ is a variety, we have a restriction map

$$k[x_1, \ldots, x_n] \twoheadrightarrow \Gamma(V)$$

whose kernel is precisely the functions that vanish on V, i.e. $I(V)$. So we have...

Prop / Def: $\Gamma(V) \cong k[x_1, \ldots, x_n] / I(V)$

Remark: $\Gamma(V)$ is ring-finite over k, and if V is a variety, since $I(V)$ is prime, $\Gamma(V)$ is an integral domain.

Def: A subvariety of V is a variety $W \subseteq \mathbb{A}^n$ s.t. $W \subseteq V$.

Thus, $\left\{\begin{matrix}\text{subvarieties} \\ \text{of } V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals} \\ \text{in } \Gamma(V)\end{matrix}\right\}$

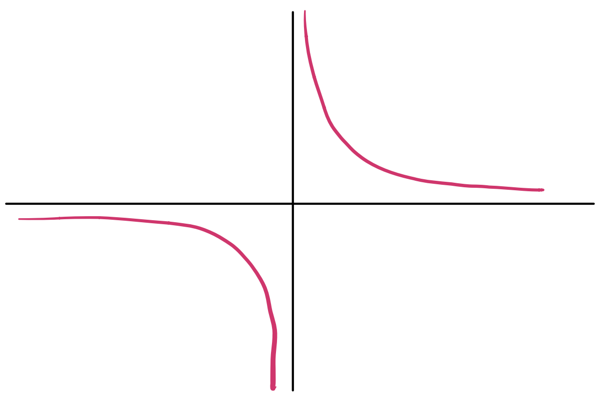
$\left\{\begin{matrix}\text{points of} \\ V\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{maximal ideals} \\ \text{in } \Gamma(V)\end{matrix}\right\}$

We can define the function $\Gamma(V) \twoheadrightarrow \Gamma(W)$ to be the restriction map $\bar{f} \longmapsto \bar{f}|_W$, where $\bar{f} \in k[x_1,\dots,x_n]/I(V)$, $f \in k[x_1,\dots,x_n] = R$

$\bar{f}$ is in the kernel $\Leftrightarrow$ $\bar{f}$ vanishes on $W \Leftrightarrow \bar{f} \in \underbrace{I_V(W)}_{\substack{\text{the ideal in} \\ \Gamma(V) \text{ corr. to } W}}$

So $\Gamma(W) \cong \Gamma(V)/I_V(W) \cong \left(R/I(V)\right) \Big/ \left(I(W)/I(V)\right)$

Ex: Going back to $V = V(xy-1) \subseteq \mathbb{A}^2$



As we saw, $y = \frac{1}{x}$ in $\Gamma(V)$, so

$\Gamma(V) = k[x,y]/(xy-1) \cong k[x, \frac{1}{x}]$, i.e.

Laurent polynomials.